DET KGL. DANSKE VIDENSKABERNES SELSKAB
MATEMATISK-FYSISKE MEDDELELSER, BIND XXII, NR. 19

GENERAL PROPERTIES OF THE CHARACTERISTIC MATRIX IN THE THEORY OF ELEMENTARY PARTICLES II

BY

C. MØLLER

KØBENHAVN
I KOMMISSION HOS EJNAR MUNKSGAARD
1946

Printed in Denmark
Bianco Lunos Bogtrykkeri A/S

INTRODUCTION

In the first paper[1] (in the following quoted as I) we especially considered those general properties of HEISENBERG's characteristic matrix S wich, in all essentials, are a consequence of the connection between the matrix elements of S and the cross sections for all kinds of collisions between elementary particles. In the present paper we shall in the first place treat the question how the discrete energy values in closed stationary states of a system of elementary particles are determined by the characteristic matrix. In I it was shown that new fundamental assumptions regarding the characteristic matrix are necessary for the solution of this problem.

Besides the energy values of closed stationary states there are, however, another group of quantities which are so closely connected with the experimental data obtained in laboratories that they must be considered "observable" in any theory, viz. the decay constants for systems of particles which can emit one of the particles. An α-radioactive nucleus represents a typical case of this kind. If, as claimed by HEISENBERG, the characteristic matrix is to give a complete description of all "observable" quantities for any atomic system, the energies of the particles emitted in a radioactive process as well as the decay constants of the systems must be derivable from the characteristic matrix of the system.

The clue to the solution of these problems was given by KRAMERS[2], who remarked that the Schrödinger wave function ψ_{W^0} belonging to a continuous energy value W^0 in all physically important cases in ordinary quantum mechanics is an analytic function of the variable W^0. By the process of analytic continuation ψ_{W^0} may then be given an unambiguous meaning also for complex values of the variable W^0 as well as

for real values W^0 smaller than the minimum value W_m^0 of the energy in a continuous state. In any case ψ_{W^0} will be a solution of the Schrödinger equation but it is not possible to give a physical interpretation of this solution for all values of W^0. Consider, for instance, the case of a real $W^0 < W_m^0$; then the asymptotic expression of ψ_{W^0} for large values of the relative distance between the particles consists of two terms, the first of which vanishes for large distances while the other increases exponentially with increasing distance. This last term contains an eigenvalue S^0 of the characteristic matrix as factor. Thus ψ_{W^0} will be an eigenfunction corresponding to a closed stationary state only for those real values of $W^0 < W_m^0$ which make S^0 equal to zero.

Therefore, in order that the energy values in closed stationary states are to be derivable from the characteristic matrix, we must assume that the characteristic matrix in all cases is an analytic function of the total kinetic energy W of the system. The energy values of the closed stationary states are then simply given by the zero points of the eigenvalue of S on the real axis in the complex W^0-plane, the eigenvalue W^0 of W being regarded as a complex variable.

In the first section of the present paper the extension of the ordinary quantum mechanical transformation theory to the case of complex eigenvalues of W is treated in detail. In section 2 the asymptotic expression for the wave functions of a two particle system in the case of complex values of W^0 is explicitly written down, and in section 3 the "observable" quantities of the discrete stationary systems are derived from the eigenvalues of the characteristic matrix. Since S is an invariant matrix, the treatment of the discrete states may be performed in any Lorentz frame of reference with equal ease. In section 4 a new general condition regarding the character of the zero points of S^0 is derived, and the question is discussed whether it is possible from a given characteristic matrix S to construct a Hamiltonian which by means of the Schrödinger equation gives the same values for the cross sections and for the discrete energy values as those derived from the matrix S.

In section 5 it is shown that also the energy E and the decay constant λ of a radioactive system may be calculated

from the eigenvalues of the characteristic matrix. In fact, these quantities are given as the real and imaginary parts, respectively, of those complex values $W^0 = E - i\frac{\lambda}{2}$ of W^0 in the lower half plane for which $S^0 = S(W^0)$ is infinite. Finally, in the last section, we shall treat a few simple familiar atomic systems by means of the characteristic matrix in order to illustrate the general theory.

It thus seems that all experimental results may be described by means of HEISENBERG's characteristic matrix without making use of the wave functions of ordinary quantum mechanics, and the way is open for a relativistic description of atomic phenomena which does not involve the difficulties inherent in all relativistic quantum field theories of the Hamiltonian form.

---

1. On the Use of Complex Variables in Quantum Mechanics.

In the first place we shall treat a simple system of two spinless particles with the rest mass κ according to quantum mechanics. If we introduce the total momentum $\boldsymbol{K}$ and the "relative" momentum $\boldsymbol{k}$ defined by

$$\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2, \quad \boldsymbol{k} = \frac{\boldsymbol{k}_2 - \boldsymbol{k}_1}{2}, \tag{1}$$

we get for the total kinetic energy

$$W = W_1 + W_2 =$$

$$= \sqrt{\kappa^2 + \left|\frac{1}{2}\boldsymbol{K} - k\boldsymbol{n}\right|^2} + \sqrt{\kappa^2 + \left|\frac{1}{2}\boldsymbol{K} + k\boldsymbol{n}\right|^2}, \tag{2}$$

where

$$\boldsymbol{n} = \frac{\boldsymbol{k}}{k} \tag{3}$$

is a unit vector in the direction of the relative momentum $\boldsymbol{k}$. For a given value $\boldsymbol{K}'$ of the total momentum the eigenvalue

W' of the total kinetic energy may take on all real values of the interval

$$W'_m < W' < \infty, \tag{4}$$

where

$$W'_m = \sqrt{(2\kappa)^2 + |\boldsymbol{K}'|^2} \tag{5}$$

represents the minimum value of W attained for $k = 0$.

From (2) we get

$$\left.\begin{aligned} \frac{\partial W}{\partial k} &= \frac{k - \frac{1}{2}(\boldsymbol{K}\boldsymbol{n})}{\sqrt{\kappa^2 + \left|\frac{1}{2}\boldsymbol{K} - k\boldsymbol{n}\right|^2}} + \frac{k + \frac{1}{2}(\boldsymbol{K}\boldsymbol{n})}{\sqrt{\kappa^2 + \left|\frac{1}{2}\boldsymbol{K} + k\boldsymbol{n}\right|^2}} = \\ &= \frac{\boldsymbol{k}_2\boldsymbol{n}}{W_2} - \frac{\boldsymbol{k}_1\boldsymbol{n}}{W_1} = (\boldsymbol{v}_2 - \boldsymbol{v}_1)\boldsymbol{n}, \end{aligned}\right\} \tag{6}$$

which is zero for $k = 0$, but positive for any other value of $k > 0$. Further, by solving the equation (2) with respect to the variable k, we get,

$$k = \frac{W}{2}\sqrt{\frac{W^2 - (2\kappa)^2 - |\boldsymbol{K}|^2}{W^2 - (\boldsymbol{K}\boldsymbol{n})^2}}. \tag{7}$$

If $(\boldsymbol{k}'_1\,\boldsymbol{k}'_2\,|)$ and $(\boldsymbol{K}'\boldsymbol{k}'\,|)$ are the representatives of the same state in two representations where the variables $(\boldsymbol{k}_1, \boldsymbol{k}_2)$ and $(\boldsymbol{K}, \boldsymbol{k})$ are on diagonal form, respectively, we have simply

$$(\boldsymbol{k}'_1\,\boldsymbol{k}'_2\,|) = (\boldsymbol{K}'\boldsymbol{k}'\,|), \tag{8}$$

since the functional determinant $\dfrac{\partial(\boldsymbol{K}', \boldsymbol{k}')}{\partial(\boldsymbol{k}'_1, \boldsymbol{k}'_2)} = 1$. Furthermore, if we introduce the variables $(x) = (\zeta, \varphi)$ defined by the equations

$$\left.\begin{aligned} &(x) = (\zeta, \varphi) \\ &\zeta = \cos\theta = n_z, \quad \varphi = \operatorname{arctg}\frac{n_y}{n_x}, \end{aligned}\right\} \tag{9}$$

we get for the representative of the state in a $(\boldsymbol{K}, W, x)$-representation

$$(\boldsymbol{K}'\boldsymbol{k}'\,|) = \sqrt{J'}\,(\boldsymbol{K}'W'x'\,|), \tag{10}$$

where the positive functional determinant is given by

$$\mathcal{A}' = \frac{\partial(\boldsymbol{K}', W', x')}{\partial(\boldsymbol{K}', \boldsymbol{k}')} = \frac{(\boldsymbol{v}_2' - \boldsymbol{v}_1')\,\boldsymbol{n}'}{k'^2} = \frac{1}{k'^2}\frac{\partial W'}{\partial k'} \tag{11}$$

on account of (6).

Now, if $\boldsymbol{X}$ and $\boldsymbol{x}$ denote the coordinates of the centre of gravity and the relative coordinates, respectively, we have

$$\left.\begin{aligned} \boldsymbol{X} &= \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2} \\ \boldsymbol{x} &= \boldsymbol{x}_2 - \boldsymbol{x}_1, \end{aligned}\right\} \tag{12}$$

and, since the functional determinant $\dfrac{\partial(\boldsymbol{X}, \boldsymbol{x})}{\partial(\boldsymbol{x}_1, \boldsymbol{x}_2)} = 1$, the transformation function connecting the $(\boldsymbol{X}, \boldsymbol{x})$-representation with the $(\boldsymbol{K}, \boldsymbol{k})$-representation is identical with the function (I, 80) expressed in terms of the new variables. However, the relative coordinate vector $\boldsymbol{x}$ commutes with $\boldsymbol{K}$, we may therefore also use a $(\boldsymbol{K}, \boldsymbol{x})$-representation and, for a suitable choice of the phases in the $(\boldsymbol{K}, W, x)$-representation, the transformation function connecting these two representations is simply*

$$(\boldsymbol{K}''\,\boldsymbol{x}'' \,|\, \boldsymbol{K}'\,W'\,x') = (2\pi)^{-\frac{3}{2}}\,\delta(\boldsymbol{K}'' - \boldsymbol{K}')\,\frac{e^{i(\boldsymbol{x}''\,\boldsymbol{n}')\,k'}}{\sqrt{\mathcal{A}'}}. \tag{13}$$

From now on, we shall have to do exclusively with states where $\boldsymbol{K}$ has a well-defined value $\boldsymbol{K}^0$. All wave-functions will thus contain a factor $\delta(\boldsymbol{K}' - \boldsymbol{K}^0)$, which will be omitted in the following. In the same way we shall omit the factor $\delta(\boldsymbol{K}' - \boldsymbol{K}^0)$ occurring in all matrices like Ψ, T, U, V (cf. I, 16 . In the $(\boldsymbol{K}, W, x)$-representation the wave matrix Ψ, defined by the equations (10), (15), and (16) in I, then takes the form

$$\left.\begin{aligned} &(W'\,x' \,|\, \Psi \,|\, W^0\,x^0) = \\ &= \delta(W' - W^0)\,\delta(x' - x^0) + \delta_+(W' - W^0)\,(W'\,x' \,|\, U \,|\, W^0\,x^0) \end{aligned}\right\} \tag{14}$$

with

* The variables x and $\boldsymbol{x}$ should not be confused! While x denotes the angle variables (9) determining the direction of the relative momentum, the heavily printed $\boldsymbol{x}$ throughout this paper denotes the coordinate vector in the relative configuration space.

$$(W'x'\,|\,U\,|\,W^0x^0) = (W'x'\,|\,U_{\boldsymbol{K}^0}\,|\,W^0x^0).$$

The function (14) is a solution of the Schrödinger equation (I, 7).

If $\alpha = (\boldsymbol{K}, W, \beta)$ is a complete set of collision constants, the transformation function connecting the $(\boldsymbol{K}, W, x)$-representation with a $(\boldsymbol{K}, W, \beta)$-representation is of the form

$$(W''x'\,|\,W^0\beta^0) = \delta(W'' - W^0)(x'\,|\,\beta^0), \tag{15}$$

where

$$(x'\,|\,\beta^0) = (x'\,|\,\beta^0)_{\boldsymbol{K}^0, W^0} \tag{16}$$

also depends on $\boldsymbol{K}^0$ and W^0. In a mixed representation we get for the representative of the wave matrix Ψ

$$\left.\begin{aligned}&(W'x'\,|\,\Psi\,|\,W^0\beta^0) = \\ &= \delta(W' - W^0)(x'\,|\,\beta^0) + \delta_+(W' - W^0)(W'x'\,|\,U\,|\,W^0\beta^0).\end{aligned}\right\} \tag{17}$$

On account of the equation

$$\delta_+(W' - W^0) + \delta_-(W'' - W^0) = \delta(W' - W^0), \tag{18}$$

following from (I, 14), this may be written

$$\left.\begin{aligned}&(W'x'\,|\,\Psi\,|\,W^0\beta^0) = \\ &= \delta_-(W' - W^0)(x'\,|\,\beta^0) + \delta_+(W' - W^0)(W'x\,|\,A\,|\,W^0\beta^0),\end{aligned}\right\} \tag{19}$$

with

$$(W'x'\,|\,A\,|\,W^0\beta^0) = (x'\,|\,\beta^0) + (W'x'\,|\,U\,|\,W^0\beta^0).$$

By means of the equations (23), (49), and (26) in I we get for $W'' = W^0$

$$\left.\begin{aligned}(W^0x'\,|\,A\,|\,W^0\beta^0) &= \int[\delta(x' - x^0) + (x'\,|\,R\,|\,x^0)]\,dx^0\,(x^0\,|\,\beta^0) \\ &= \int(x'\,|\,S\,|\,x^0)\,dx^0\,(x^0\,|\,\beta^0) = S^0(x'\,|\,\beta^0),\end{aligned}\right\} \tag{20}$$

where S^0 is the eigenvalue of S, corresponding to the values $\alpha^0 = (\boldsymbol{K}^0, W^0, \beta^0)$ for the collision constants in the complete set.

When the equation (I, 7) is written in a mixed representation it is seen that the function

$$\psi_{\alpha^0}(W'x') = \psi_{W^0\beta^0}(W'x') = (W'x' \mid \psi \mid W^0\beta^0) \qquad (21)$$

is a solution of the Schrödinger equation

$$\left.\begin{aligned}(W^0 - W')\,\psi_{W^0\beta^0}(W'x') &= \\ = \int (W'x' \mid V \mid W''x'')\,dW''\,dx''\,\psi_{W^0\beta^0}(W''x'')\end{aligned}\right\} (22)$$

for all real values of W^0 in the interval

$$\left.\begin{aligned}&W^0_m < W^0 < \infty \\ &W^0_m = \sqrt{(2K)^2 + |\boldsymbol{K}^0|^2}.\end{aligned}\right\} (23)$$

Until now, the variables W^0, W', W'' as eigenvalues of the total kinetic energy have been considered real quantities which could take on all values of the interval (23). In what follows, this "original interval" will be denoted by the symbol J_1. Following KRAMERS' idea we shall now consider solutions of the Schrödinger equation corresponding to a complex value of the energy W^0. If all the functions occurring in (22) are analytic functions of the variables W^0, W', W'', they will have a meaning also in a certain region outside the original interval and all integral relations, such as the "Schrödinger equation" (22), will hold also in this extended region. From now on, the "eigenvalues" W^0, W', W'' of W will be regarded as complex variables, while the eigenvalues of $\boldsymbol{K}$, x, $\beta \cdots$ in general are real variables as before.

Now, we shall first give a meaning to the functions $\delta_{\pm}(W' - W^0)$ occurring in (14) for complex values of W' and W^0. In the original interval J_1 these functions are defined by (I, 14), and the integral $\int f(W')\,\delta_{\pm}(W' - W^0)\,dW'$ is understood to mean the Cauchy principle value of the integral extended from W^0_m to ∞. If $f(W')$ is an analytic function these integrals, as mentioned by HEISENBERG, are equal to the complex integrals

$$\int_{C_{\pm}(W^0)} \frac{\pm f(W')\,dW'}{2\pi i\,(W' - W^0)},$$

where the paths of integration $C_+(W^0)$ and $C_-(W^0)$ are two curves consisting of the portions of the real axis joining the

points $(W_m^0, W^0 - \varepsilon)$ and $(W^0 + \varepsilon, +\infty)$ and of small semi-circles with centre in W^0 and radius ε below and above the real axis, respectively. The curves $C_\pm(W^0)$ may, of course, be arbitrarily deformed inside the region of the W'-plane, where the integrand is analytic.

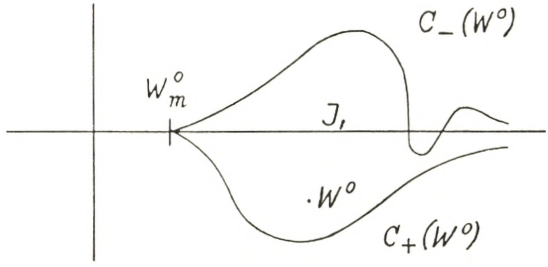

Fig. 1.

If W^0 has a complex value, we may now *define* the functions $\delta_\pm(W' - W^0)$ by the equations

$$\int f(W')\,\delta_\pm(W' - W^0)\,dW' = \pm\int_{C_\pm(W^0)} \frac{f(W')\,dW'}{2\pi i\,(W' - W^0)}, \qquad (24)$$

$$\int \delta_\pm(W^0 - W')\,f(W')\,dW' = \pm\int_{C_\mp(W^0)} \frac{f(W')\,dW'}{2\pi i\,(W^0 - W')}, \qquad (25)$$

where the paths of integration $C_+(W^0)$ and $C_-(W^0)$ are two curves connecting the points W_m^0 and ∞ in such a way that W^0 lies to the left of $C_+(W^0)$ but to the right of $C_-(W^0)$. Further, C_+ and C_- must be chosen such that $f(W')$ is analytic at all points on the curves and inside the region bounded by the curves $C_\pm$ and J_1 (see Fig. 1). Strictly speaking, the functions δ_+ and δ_- thus have a rigorous meaning only when they appear as a factor in an integrand, just as in the case of real W' and W^0. For $W^0 \prec J_1$ (W^0 inside J_1) the equations (24) and (25) are easily seen to be in accordance with the definitions (I, 14).

By means of (24), the equation (25) may be written

$$\int \delta_\pm(W^0 - W')\,f(W')\,dW' = \mp\int_{C_\mp(W^0)} \frac{f(W')\,dW'}{2\pi i\,(W' - W^0)} = \int f(W')\,\delta_\mp(W' - W^0)\,dW'. \qquad (26)$$

This result, which holds for an arbitrary (analytic) function f, can be expressed by the equation

$$\delta_{\pm}(W^0 - W') = \delta_{\mp}(W' - W^0). \tag{27}$$

Further, we may define the functions $\delta(W'-W^0)$ and $\delta(W^0-W')$ by the equations

$$\left.\begin{aligned}\int f(W')\,\delta(W'-W^0)\,dW' &= \int \delta(W^0-W')\,f(W')\,dW' = \\ &= \int_{C(W^0)} \frac{f(W')\,dW'}{2\pi i\,(W'-W^0)} = f(W^0),\end{aligned}\right\} \tag{28}$$

where $C(W^0)$ is a contour encircling the point W^0 in the counter-clockwise sense in such a way that $f(W')$ is analytic at any point inside C. From (24) we then get

$$\left.\begin{aligned}\int f(W')\left[\delta_+(W'-W^0)+\delta_-(W'-W^0)\right]dW' &= \\ = \int_{C(W^0)} \frac{f(W')\,dW'}{2\pi i\,(W'-W^0)} &= \int f(W')\,\delta(W'-W^0)\,dW',\end{aligned}\right\} \tag{29}$$

a result which may be expressed by the equation

$$\delta_+(W'-W^0)+\delta_-(W'-W^0) = \delta(W'-W^0) = \delta(W^0-W'). \tag{30}$$

In the same sense we have, according to (24), (25), and (28),

$$\left.\begin{aligned}2\pi i(W'-W^0)\,\delta_+(W'-W^0) = -2\pi i(W'-W^0)\,\delta_-(W'-W^0) = 1 \\ (W'-W^0)\,\delta(W'-W^0) = 0,\end{aligned}\right\} \tag{31}$$

just as in the case of real arguments.

Now, let $f(W')$ be an arbitrary analytic function originally given in the interval J_1 and defined in a larger region Ω of the W'-plane by the process of analytic continuation. In the following, Ω will be a common notation for the regions where the functions considered are analytic. Ω may very well contain singular points where the functions have poles of any order. The extension of the region Ω will thus depend on the kind of

function considered. By the "adjoint" function $f(W')^\dagger$ we mean that (uniquely determined) analytic function, which in J_1 is identical with the conjugate complex function $f(W')^*$. Outside J_1 the functions $f^\dagger$ and f^* will not be identical and the regions Ω, where the functions f and $f^\dagger$ are analytic, generally will also not be identical. The functions f and $f^\dagger$ will be identical only if f is real in the original interval. In particular, we have

$$W'^\dagger = W'. \tag{32}$$

Let $g_\pm(W^0)$ be the functions connected with f by the equations

$$g_\pm(W^0) = \int f(W')\,\delta_\pm(W' - W^0)\,dW'. \tag{33}$$

For $W^0 \prec J_1$ we then, according to (I, 14) and (24) have,

$$g_\pm(W^0)^* = \left\{ \int_{W_m^0}^{\infty} f(W')\,\delta_\pm(W' - W^0)\,dW' \right\}^* =$$

$$= \int_{W_m^0}^{\infty} f(W')^*\,\delta_\mp(W' - W^0)\,dW' = \mp \int_{C_\mp(W^0)} \frac{f(W')^\dagger\,dW'}{2\pi i (W' - W^0)}.$$

By "analytic continuation" of this relation we thus get

$$g_\pm(W^0)^\dagger = \mp \int_{C_\mp(W^0)} \frac{f(W')^\dagger\,dW'}{2\pi i (W' - W^0)} = \int f(W')^\dagger\,\delta_\mp(W' - W^0)\,dW'. \tag{34}$$

The result of comparison of (33) and (34) may be expressed by the equations

$$\delta_\pm(W' - W^0)^\dagger = \delta_\mp(W' - W^0) = \delta_\pm(W^0 - W'), \tag{35}$$

where we have used the equations (27), also. Further, we get from (30) and (35)

$$\delta(W' - W^0)^\dagger = \delta(W' - W^0). \tag{36}$$

Consider now an arbitrary transformation function connecting two representations in the quantum mechanical transformation theory, e. g. the functions $(W'x' \mid W^0\beta^0)$ in (15). Since

$(W^0\beta^0 \mid W'x')$ is equal to $(W'x' \mid W^0\beta^0)^*$ for W' and W^0 inside J_1 we have quite generally

$$(W^0\beta^0 \mid W'x') = (W'x' \mid W^0\beta^0)^\dagger \tag{37}$$

for W' and W^0 inside Ω. All integral relations between these functions which hold inside J_1 will hold also inside the wider region Ω, i. e. we have e. g.

$$\int (W'\beta' \mid W''x'')\, dW''dx'' (W''x'' \mid W^0\beta^0) = \delta(W'-W^0)\,\delta(\beta'-\beta^0), \tag{38}$$

or by (15),

$$\int (\beta' \mid x'')_{W^0}\, dx'' (x'' \mid \beta^0)_{W^0} = \delta(\beta'-\beta^0). \tag{39}$$

If $(W'x' \mid A \mid W^0x^0)$ denotes the representative of an operator A in the (W, x)-representation, this function is defined for all values of W' and W^0 inside J_1. Supposing that this function is an analytic function of W' and W^0, the "matrix elements" $(W'x' \mid A \mid W^0x^0)$ may, by the process of analytic continuation, be defined in a larger region Ω comprising also complex values of W' and W^0. We now define the "adjoint" matrix $A^\dagger$ by the equation

$$(W^0x^0 \mid A^\dagger \mid W'x') = (W'x' \mid A \mid W^0x^0)^\dagger. \tag{40}$$

For W' and W^0 inside J_1 the equation (40) is identical with the ordinary definition (I, 17) of the Hermitian conjugate matrix $A^\dagger$. For matrices of the form (I, 15) and (I, 30), i. e.

$$(W'x' \mid T_\pm \mid W^0x^0) = \delta_\pm (W'-W^0)\,(W'x' \mid U_\pm \mid W^0x^0), \tag{41}$$

we get, by (40) and (35),

$$(W'x' \mid T_\pm^\dagger \mid W^0x^0) = \delta_\pm (W'-W^0)\,(W'x' \mid U_\pm^\dagger \mid W^0\beta^0). \tag{42}$$

The ordinary rule for "matrix multiplication"

$$(W'x'|AB|W^0x^0) = \int (W'x'|A|W''x'')\, dW''\, dx'' (W''x''|B|W^0x^0) \tag{43}$$

may be extended to the case of complex values of the variables W. The path of integration in $\int dW''$ may then be arbitrarily deformed inside the region Ω. From (40) and (43) we get

$$\left.\begin{aligned}(W^0x^0\,|\,(AB)^\dagger\,|\,W'x') &= (W'x'\,|\,AB\,|\,W^0x^0)^\dagger = \\ &= \int (W'x'\,|\,A\,|\,W''x'')^\dagger\, dW''dx''\,(W''x''\,|\,B\,|\,W^0x^0)^\dagger = \\ &= \int (W^0x^0\,|\,B^\dagger\,|\,W''x'')\, dW''dx''\,(W''x''\,|\,A^\dagger\,|\,W'x') = \\ &= (W^0x^0\,|\,B^\dagger A^\dagger\,|\,W'x').\end{aligned}\right\} \quad (44)$$

Thus, the ordinary matrix rule

$$(AB)^\dagger = B^\dagger A^\dagger \quad (45)$$

is seen to hold also for the generalized matrices with complex values of the W-variables. From (34) or (35) and (42) it follows that the equation (44) is true also in the case where one of the functions, say $(W'x'|B|W^0x^0)$, contains a factor $\delta_\pm(W'-W^0)$ like the matrix $T_\pm$ in (41).

If A is a Hermitian matrix in the original interval the equation

$$A^\dagger = A \quad (46)$$

will hold also in the larger region Ω, i. e. A is a self-adjoint matrix. Similarly, if A is a unitary matrix in the original sense the generalized matrix will also satisfy the generalized equations

$$A^\dagger A = AA^\dagger = 1. \quad (47)$$

Thus the matrices η and S, connected by the equation (I, 43), will satisfy the generalized equations (46) and (47), respectively. In a representation where the collision constants (W, β) are diagonal we have

$$\left.\begin{aligned}(W'\beta'\,|\,\eta\,|\,W^0\beta^0) &= \eta^0\,\delta(W'-W^0)\,\delta(\beta'-\beta^0) \\ (W'\beta'\,|\,S\,|\,W^0\beta^0) &= S^0\,\delta(W'-W^0)\,\delta(\beta'-\beta^0).\end{aligned}\right\} \quad (48)$$

The "eigenvalues" η^0, S^0 are functions of the variables (W^0, β^0) and they will satisfy the equations

$$\left.\begin{aligned}\eta^{0\dagger} &= \eta^0 \\ S^{0\dagger} &= \frac{1}{S^0},\end{aligned}\right\} \quad (49)$$

respectively. All matrix equations derived in I for the original region also hold in the wider region Ω, provided, of course, that the representative of the potential energy V in the (W, x)-representation is an analytic function of the energy-variables in the original interval. This restriction is, however, not serious. It does not mean that the potential function in *configuration space* must be an analytic function or even a continuous function of the position coordinates. From FOURIER's theorem it follows that, if the potential vanishes sufficiently rapidly with increasing distance of the particles, the representative of the potential energy V in momentum space will always be an analytic function of the momentum variables.

2. The Asymptotic Form of the Wave Functions.

We shall now consider the wave function $\psi_{\alpha^0}(\boldsymbol{x}')$ in a $(\boldsymbol{K}, \boldsymbol{x})$-representation where $\boldsymbol{x}$ is the relative coordinate vector defined by (12). Omitting again the factor $\delta(\boldsymbol{K}' - \boldsymbol{K}^0)$, we get from (13) and (19)

$$\left.\begin{aligned} &\psi_{\alpha^0}(\boldsymbol{x}') = \psi_{W^0\beta^0}(\boldsymbol{x}') = (\boldsymbol{x}' \,|\, \psi \,|\, W^0\,\beta^0) = \\ &= (2\pi)^{-\frac{3}{2}} \left\{ \int \frac{e^{i(\boldsymbol{x}'\boldsymbol{n}')k'}}{\sqrt{A'}}\, dW'\, dx'\, \delta_-(W' - W^0)\,(x' \,|\, \beta^0) \right. \\ &\left. + \int \frac{e^{i(\boldsymbol{x}'\boldsymbol{n}')k'}}{\sqrt{A'}}\, dW'\, dx'\, \delta_+(W' - W^0)\,(W'x' \,|\, A \,|\, W^0\,\beta^0), \right\} \end{aligned}\right\} \quad (50)$$

where $\boldsymbol{n}' = \boldsymbol{n}(x')$ is the unit-vector corresponding to the values (x') of the variables (9). Further, $k' = k(\boldsymbol{K}^0, W', x')$ is the value of k obtained from (7) by putting $\boldsymbol{K} = \boldsymbol{K}^0$, $\boldsymbol{n} = \boldsymbol{n}'$, and $W = W'$, and $A' = A(\boldsymbol{K}^0, W', x')$ is the corresponding value of the functional determinant (11). Thus, using (23), we have

$$k' = \frac{W'}{2} \sqrt{\frac{W'^2 - W_m^{02}}{W'^2 - (\boldsymbol{K}^0\boldsymbol{n}')^2}}. \quad (51)$$

On account of the square root occurring in the expression for k' we shall have to make a cut in the W'-plane along a

suitable curve connecting the singular point W_m and $(\boldsymbol{K}^0\boldsymbol{n}')$ on the real axis in order to make k' analytic and one-valued throughout the cut plane. Since we are particularly interested in the real values of $W' < W_m^0$, we cannot make the cut along

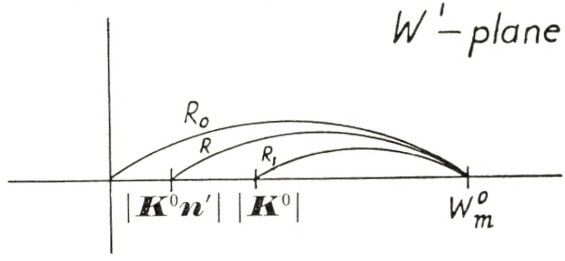

Fig. 2.

the real axis, but we can make it as close to the real axis, as we like. Putting $U = \dfrac{W'^2 - W_m^{02}}{W'^2 - (\boldsymbol{K}^0\boldsymbol{n}')^2}$ we shall define the square root of U by the equation

$$\sqrt{U} = \sqrt{\varrho}\, e^{i\frac{\varphi}{2}} \tag{52}$$

if

$$\left.\begin{aligned} U &= \varrho\, e^{i\varphi} \\ -\pi - \varepsilon < \varphi &< \pi - \varepsilon . \end{aligned}\right\} \tag{53}$$

Here ε is a finite positive number which may be chosen as small as we like. This definition corresponds to a cut along the radius vector $\varphi = \pi - \varepsilon$ in the U-plane. The corresponding cut in the W'-plane is easily seen to have the form of the curve R in Fig. 2.

Now, let R_0 and R_1 be the R-curves corresponding to $|\boldsymbol{K}^0\boldsymbol{n}'| = 0$ and $|\boldsymbol{K}^0\boldsymbol{n}'| = |\boldsymbol{K}^0|$, respectively. Outside the region ω circumscribed by the curves R_0, R_1 and the part of the real axis joining the points 0 and $|\boldsymbol{K}^0|$, k' will thus, for any value of $\boldsymbol{n}'$, be an analytic and one-valued function of W', and we shall in what follows consider such values of W', only. If ε is made sufficiently small, the forbidden region ω can be made as small as we like. Also the functions A' and $\sqrt{A'}$ are then seen to be analytic functions of W' outside ω.

We shall now calculate the asymptotic value of the function $\psi_{W^0_{\beta^0}}(\boldsymbol{x}')$ in (50) for large values of the relative distance

$r' = |\boldsymbol{x}'| = |\boldsymbol{x}'_2 - \boldsymbol{x}'_1|$. If we introduce the unit vector $\boldsymbol{e}' = \frac{\boldsymbol{x}'}{r}$, the integrals in (50) are of the form

$$X = \int e^{ir'k'(\boldsymbol{e}'\boldsymbol{n}')} f(W', x')\, dW' dx'. \tag{54}$$

For $r' \to \infty$ we get, if terms of higher order in $\frac{1}{r'}$ than the first are neglected,

$$X = \frac{2\pi}{ir'} \left\{ \int \frac{e^{ir'k'}}{k'} f(W', x')\, dW' - \int \frac{e^{-ir'k'}}{k'} f(W', x'_-)\, dW' \right\}, \tag{55}$$

where $(x') = (\zeta', \varphi')$ are the values of the variables (9) corresponding to the direction $\boldsymbol{n}' = \boldsymbol{e}'$, while $(x'_-) = (-\zeta', \varphi' + \pi)$ correspond to the opposite direction $\boldsymbol{n}' = -\boldsymbol{e}'$. Similarly, k' in (55) is the value of k' in (51) for $\boldsymbol{n}' = \pm \boldsymbol{e}'$. The equation (55) follows at once by a partial integration if we temporarily introduce (k', x') instead of (W', x') as integration variables.

Since the functions f in our case contain a factor $\delta_\pm (W' - W^0)$, we have to calculate for $r' \to \infty$ integrals of the type

$$Y_\pm = \int e^{ir'k'} \delta_\pm (W' - W^0)\, g(W')\, dW' = \int_{C_\pm(W^0)} \frac{e^{ir'k'} g(W')\, dW'}{\pm 2\pi i (W' - W^0)}, \tag{56}$$

$$Z_\pm = \int e^{-ir'k'} \delta_\pm (W' - W^0)\, g(W')\, dW' = \int \frac{e^{-ir'k'} g(W')\, dW'}{\pm 2\pi i (W' - W^0)}. \tag{57}$$

Now, let $R(z)$ and $I(z)$ denote the real and imaginary parts of a complex number z, respectively. From the definitions (51), (52), and (53) it then follows immediately that the imaginary part of k' is positive for $R(W') > 0$ if W' lies above the real axis and outside ω. Further, we have $I(k') < 0$ for $R(W') > 0$ if W' lies below the forbidden region ω. Let Ω_+ and Ω_- be those parts of the region Ω for which $I(k')$ is positive and negative, respectively. For $W' \prec \Omega_+$ the exponential functions in (56) contain a factor $e^{-\alpha r'}$, where α is real and positive. Those parts of the paths of integration in (56) which lie inside Ω_+ will thus give a vanishing contribution to $Y_\pm$ in the limit $r' \to \infty$.* Similarly, those parts of the integration curves which lie inside Ω_- will give a vanishing contribution to $Z_\pm$ for

* Provided the interchange of the limiting process $r' \to \infty$ and the operation of integration is allowed.

$r' \to \infty$. If we choose the curves C_+ and C_- as in Fig. 3, we get for $r' \to \infty$ the asymptotic expressions

$$Y_\pm = \left\{ \begin{array}{l} \int\limits_{C(W^0)} \dfrac{e^{ir'k'}\, g(W')\, dW'}{2\pi i (W' - W^0)} = e^{ir'k^0} g(W^0) \\ \\ 0, \end{array} \right\} \quad (58)$$

where k^0 denotes the value of k' following from the 'theorem of conservation of energy', i. e.

$$k^0 = \frac{W^0}{2} \sqrt{\frac{W^{0^2} - W_m^{0^2}}{W^{0^2} - (\boldsymbol{K}^0 \boldsymbol{e}')^2}} \quad (59)$$

By a suitable choice of the curves C_+ and C_- in (57) we similarly get the asymptotic expressions

$$Z_\pm = \left\{ \begin{array}{l} 0 \\ e^{-ir'k^0} g(W^0). \end{array} \right\} \quad (60)$$

Thus, by means of the equations (54)—(60) and by (20), we obtain the following asymptotic expression for the wave function (50) in the limit $r' \to \infty$

$$\left. \begin{array}{l} \Psi_{W^0 \beta^0}(\boldsymbol{x}') = (\boldsymbol{x}' \,|\, \Psi \,|\, W^0 \beta^0) = \\ \\ = \dfrac{(2\pi)^{-\frac{1}{2}}}{ir'k^0 \sqrt{\Delta^0}} \left\{ -e^{-ir'k^0}(x'_- \,|\, \beta^0) + e^{ir'k^0} S^0(x' \,|\, \beta^0) \right\} \end{array} \right\} \quad (61)$$

with

$$\Delta^0 = \Delta(\boldsymbol{K}^0, W^0, x') = \Delta(\boldsymbol{K}^0, W^0, x'_-). \quad (62)$$

For a system consisting of two particles only, we may take for the collision constants β the variables L and m defined by (173), (183), (163), and (160) in I, and in the following sections the symbol β is used simply as an abbreviation for the two quantities L and m, which have the discrete eigenvalues $L^0 = l^0(l^0 + l)$, $l^0 = 0, 1, 2 \cdots$ and $m^0 = -l^0, -l^0 + 1, \cdots + l^0$, respectively. Since I^{μ} is a pseudo-four-vector the variables L and m are invariant under spatial reflections at the origin. However, L and m are invariant also under the transformation $(x') \to (x'_-)$ or $\boldsymbol{k}' \to -\boldsymbol{k}'$, which corresponds to a reflection in the relative space only; for this transformation is equivalent to a permutation of the two particles, as is seen

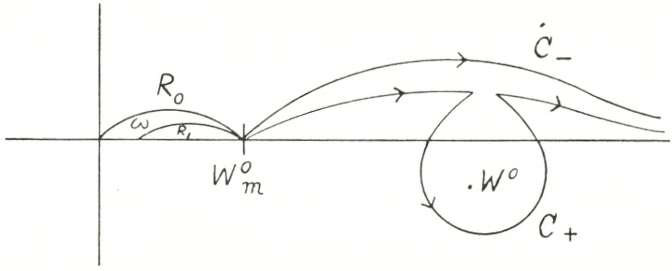

Fig. 3.

from (1), and the variables L and m are symmetrical in the particles. Thus, if τ denotes the operator corresponding to this transformation, τ commutes with L and m and the eigenfunctions $(x' \mid l^0 m^0)$ of L and m will also be eigenfunctions of τ. Now, since $\tau^2 = 1$, the eigenvalues of τ are ± 1, hence we get

$$\tau (x' \mid l^0 m^0) = (x'_- \mid l^0 m^0) = \pm (x' \mid l^0 m^0). \tag{63}$$

In the centre of gravity system the variables L and m are identical with M^2 and M_z, and the functions $(x' \mid l^0 m^0)$ are the usual spherical harmonics $Y_{l^0 m^0}(\zeta', \varphi')$. Thus we have in this frame of reference

$$(x'_- \mid l^0 m^0) = (-1)^{l^0} (x' \mid l^0 m^0). \tag{64}$$

However, since τ commutes with the variables $\boldsymbol{N}$ and $\boldsymbol{M}$ defined by (132) and (139) in I, τ is a relativistic invariant on account of (I, 135) and (I, 138), and the equations (64) will, therefore, hold in all frames of reference.

The asymptotic expression (61) for the wave function thus takes the form

$$\left.\begin{aligned} \psi_{\alpha^0}(\boldsymbol{x}') &= \psi_{W^0 l^0 m^0}(\boldsymbol{x}') = \\ &= \frac{(2\pi)^{-\frac{1}{2}}}{ir' k^0 \sqrt{A^0}} \left\{ e^{-ir' k^0} (-1)^{l^0 + 1} + e^{ir' k^0} S^0 \right\} (x' \mid l^0 m^0), \end{aligned}\right\} \tag{65}$$

where $S^0 = S(\alpha^0)$, according to (I, 185), is a function of $K^0 = \sqrt{W^{0\,2} - |\boldsymbol{K}^0|^2}$ and l^0 only:

$$S^0 = S(K^0, l^0). \tag{66}$$

$\psi_{\alpha^0}(\boldsymbol{x}')$ is a solution of the Schrödinger equation, i.e. $\psi_{\alpha^0}(\boldsymbol{x}')$ is an eigenfunction of the Hamiltonian H. However, since H does not commute with the variables L and m, $\psi_{\alpha^0}(\boldsymbol{x}')$ is not

in general an eigenfunction of L and M. This is seen also from the asymptotic expression (65), since k^0 and Δ^0 occurring in the factor of the eigenfunction $(x' \mid l^0 m^0)$ depend on (x'). Nevertheless, the eigenvalues l^0 and m^0 of the collision constants L and m may be used for labelling the eigenfunctions of H instead of the eigenvalues of a set of constants of motion which are customarily used for this purpose.

Only in the centre of gravity system, i. e. for $\boldsymbol{K}^0 = 0$, where L and m are essentially equal to M^2 and M_z, respectively, k^0 and Δ^0 will be independent of (x'), and $\Psi_{\alpha^0}(\boldsymbol{x}')$ will be an eigenfunction of L and m, also. In fact, in this case we get from (59), (23), (62), (11), and (6)

$$\left.\begin{aligned} k^0 &= \sqrt{\left(\frac{W^0}{2}\right)^2 - \kappa^2} \\ \Delta^0 &= \frac{2}{k^0 \sqrt{\kappa^2 + k^{02}}}, \end{aligned}\right\} \quad (67)$$

which are independent of (x'). The same is approximately true also in the 'non-relativistic' case where $\boldsymbol{K}^0$ may be treated as small compared with W^0.

The adjoint wave matrix $\Psi^\dagger$ satisfies the equation

$$W \Psi^\dagger - \Psi^\dagger W = \Psi^\dagger V, \quad (68)$$

as is seen from (I, 7) when it is noticed that W and V are self-adjoint matrices. If (68) is written in the mixed representation, we find that the adjoint wave function

$$\Psi^\dagger_{\alpha^0} = \Psi_{W^0\beta^0}(W'x')^\dagger = (W'x' \mid \Psi \mid W^0\beta^0)^\dagger = (W^0\beta^0 \mid \Psi^\dagger \mid W'x') \quad (69)$$

satisfies the adjoint Schrödinger equation

$$\left.\begin{aligned} &\Psi_{\alpha^0}(W'x')^\dagger (W^0 - W') = \\ &= \int \Psi_{\alpha^0}(W''x'')^\dagger \, dW'' dx'' \, (W''x'' \mid V \mid W^0x^0). \end{aligned}\right\} \quad (70)$$

In the $(\boldsymbol{K}, \boldsymbol{x})$-representation the adjoint wave function $\Psi_{\alpha^0}(\boldsymbol{x}')^\dagger$ has the asymptotic form (for $r' \to \infty$)

$$\psi_{\alpha^0}(x')^{\dagger} = \frac{(2\pi)^{-\frac{1}{2}}}{-ir'k^0\sqrt{\mathcal{A}^0}}\left\{e^{ir'k^0}(-1)^{l^0+1} + e^{-ir'k^0}S^{0\dagger}\right\}(l^0 m^0 | x'), \quad (71)$$

as is seen from (65) when we consider the fact that k^0 and $\sqrt{\mathcal{A}^0}$ are real for $W^0 \prec I$, i. e.

$$\left.\begin{aligned} k^{0\dagger} &= k^0 \\ \sqrt{\mathcal{A}^{0\dagger}} &= \sqrt{\mathcal{A}^0} \end{aligned}\right\} (72)$$

for $W^0 \prec \Omega$.

3. Determination of the Discrete Stationary States.

The function $\psi_{\alpha^0} = \psi_{W^0 l^0 m^0} = \psi_{W^0}$ is a solution of the Schrödinger equation (22) for all values of W^0 inside Ω, i. e. also for real values of W^0 in the interval J_2, defined by

$$|\boldsymbol{K}^0| < W^0 < W_m^0, \quad (73)$$

provided that the region Ω, where ψ_{W^0} is analytic, contains this interval. However, ψ_{α^0} is not an eigenfunction of H, unless ψ_{α^0} is everywhere regular, and this does not apply to all values of W^0 in J_2. From the definitions (59), (52), and (53) we get for $W^0 \prec J_2$

$$k^0 = -i|k^0| = -i\frac{W^0}{2}\sqrt{\frac{W_m^{0^2} - W^{0^2}}{W^{0^2} - (\boldsymbol{K}^0 \boldsymbol{e}')^2}}, \quad (74)$$

and the curled bracket in the asymptotic expression (65) becomes

$$e^{-|k^0|r'}(-1)^{l^0+1} + e^{|k^0|r'}S^0.$$

While the first term vanishes for $r' \to \infty$, the last term increases exponentially with r'. Hence, the condition for ψ_{α^0} being an eigenfunction is that S^0 is zero and the energy values of the system in the closed stationary states are determined as the values $W_n^0(l^0)$ of W^0 inside J_2, which make $S^0 = 0$. They are thus determined by the equations

$$\left.\begin{aligned} S^0 &= S(K_n(l^0), l^0) = 0 \\ K_n(l^0) &= \sqrt{W_n^0(l^0)^2 - |\boldsymbol{K}^0|^2}, \end{aligned}\right\} (75)$$

where n is an index labelling the zero points of S^0 corresponding to a fixed value of l^0. The quantity $K_n(l^0)$ simply denotes the value of the rest mass of the system as a whole in the stationary state considered. It has a well-defined physical meaning independently of the frame of reference.

This general result from quantum mechanics may now be assumed to hold also in cases where no Hamiltonian of the system exists. Thus, in the special case of a system of two particles we shall postulate[3] that *the characteristic matrix* $S = S(K, L)$ *is an analytic function of* K *for all values of* L and that *the values of the rest mass* K *of the system in the discrete stationary states are determined by the equation* (75). The determination of the closed stationary states in the new theory may therefore be performed in any Lorentz frame of reference with equal ease. This is a particularly beautiful feature of the new theory, which is due to the extremely simple transformation properties of the characteristic matrix.

In order to find further general conditions for the characteristic matrix we shall now return to the case where a Hamiltonian and a Schrödinger equation exist. Using (17), (16), (18), and (I, 23), (I, 26), we get for the wave function $\psi_{\alpha^0}(W'x')$ in (21)

$$\left.\begin{aligned} \psi_{\alpha^0}(W'x') = \psi_{W^0\beta^0}(W'x') = (W'x' \mid \psi \mid W^0\beta^0) = \\ = (W'x' \mid S \mid W^0\beta^0) - \delta_-(W' - W^0)(W'x' \mid U \mid W^0\beta^0). \end{aligned}\right\} \quad (76)$$

If W^0 is one of the zero values $W_n^0(l^0)$ defined by (75), we have

$$(W'x' \mid S \mid W^0\beta^0) = (W'x' \mid W^0\beta^0) \cdot S^0 = 0, \quad (77)$$

and if $r = (n, l^0, m^0)$ is an index labelling the closed stationary states, we get for the normalized eigenfunction $\psi_r(W'x')$ belonging to the discrete energy value

$$E_r = W_n^0(l^0) \quad (78)$$

the following expression:

$$\left.\begin{aligned} \Psi_r(W'x') &= N_r \cdot \Psi_{W^0 = E_r, \beta^0}(W'x') = \\ &= -N_r\, \delta_-(W' - E_r)\,(W'x' \,|\, U \,|\, r), \end{aligned}\right\} \quad (79)$$

where N_r is a normalization factor and

$$(W'x' \,|\, U \,|\, r) = (W'x' \,|\, U \,|\, W^0 = E_r, \beta^0). \qquad (79')$$

For $W' \prec J_1$ the difference $W' - E_r$ is real and positive, and we have simply

$$\Psi_r(W'x') = \frac{N_r}{2\pi i (W' - E_r)} (W'x' \,|\, U \,|\, r), \qquad (80)$$

since the interval J_1 has the character of a $C_-(E_r)$-curve.

The corresponding function $\Psi_r(\boldsymbol{x}')$ in configuration space has the asymptotic form, following from (65), (74), and (75)

$$\Psi_r(\boldsymbol{x}') = \frac{(2\pi)^{-\frac{1}{2}} N_r}{|k_r^0| \sqrt{\Delta_r^0}} (-1)^{l^0+1} \cdot \frac{e^{-|k_r^0| r'}}{r'} (x' \,|\, l^0 m^0), \qquad (81)$$

where k_r^0 and Δ_r^0 are the values of k^0 and Δ^0 for $W^0 = E_r$. The eigenfunction $\Psi_r(W'x')$ in (80) is a regular normalized solution of the Schrödinger equation (22) with $W^0 = E_r$.

While $\Psi_{W^0\beta^0}$ is regular for $W^0 = E_r$ the adjoint wave function $\Psi_{W^0\beta^0}(W'x')^\dagger$ in (69) diverges for $W^0 \to E_r$. This is seen, for instance, from the asymptotic expression (71) when account is taken of the equation (49) which shows that

$$\left.\begin{aligned} S^{0\dagger} &= \frac{1}{S^0} \to \infty \\ &\text{for} \\ W^0 &\to E_r. \end{aligned}\right\} \quad (82)$$

Let us, therefore, consider the function

$$\Phi_{\alpha^0}(W'x') = S^0\, \Psi_{\alpha^0}(W'x')^\dagger, \qquad (83)$$

which is also a solution of the adjoint Schrödinger equation (70). (The wave matrix $S\,\Psi^\dagger$ occurring in (83) is closely connected with

the matrix Ψ_- defined by the equations (29)—(31) in I). From (I, 10) and (I, 21) we get for $\Psi^\dagger$ in the mixed representation

$$\left.\begin{aligned}(W^0\beta^0\,|\,\Psi^\dagger\,|\,W'x') = \\ (W^0\beta^0\,|\,1\,|\,W'x') + \delta_+(W^0 - W')(W^0\beta^0\,|\,U^\dagger\,|\,W'x').\end{aligned}\right\}(84)$$

Hence, on account of (27),

$$\left.\begin{aligned}\Phi_{\alpha^0}(W'x') = \\ = (W^0\beta^0\,|\,S\,|\,W'x') + \delta_-(W' - W^0)(W^0\beta^0\,|\,SU^\dagger\,|\,W'x').\end{aligned}\right\}(85)$$

For $W^0 = E_r = W_n(l^0)$ the first term is zero and the function

$$\Phi_r(W'x') = \Phi_{W^0=E_r,\beta^0}(W'x') = \delta_-(W' - W^0)(r\,|\,SU^\dagger\,|\,W'x') \quad (86)$$

is a regular solution of the adjoint Schrödinger equation (70). While $(W'x'\,|\,U\,|\,W^0\beta^0)$ was regular for $W^0 \to E_r$ we see that $(W^0\beta^0|\,U^\dagger\,|\,W'x')$ diverges for $W^0 \to E_r$, since $(W^0\beta^0|\,SU^\dagger\,|\,W'x') = S^0\cdot(W^0\beta^0\,|\,U^\dagger\,|\,W'x')$ remains finite and $S^0 \to 0$ for $W^0 \to E_r$. If $W' \prec J_1$, we have simply

$$\Phi_r(W'x') = -\frac{1}{2\pi i(W' - E_r)}(r\,|\,SU^\dagger\,|\,W'x'), \qquad (87)$$

since the Interval J_1 is a $C_-(E_r)$-curve. The corresponding function $\Phi_r(\mathfrak{x}')$ in configuration space has the asymptotic form, following from (71), (49), (74), and (75),

$$\Phi_r(\mathfrak{x}') = -\frac{(2\pi)^{-\frac{1}{2}}}{|k_r^0|\sqrt{\mathcal{A}_r^0}}\cdot\frac{e^{-|k_r^0|r'}}{r'}(l^0m^0\,|\,x'). \qquad (88)$$

Now, for $W^0 = E_r$ we have

$$\left.\begin{aligned}(l^0m^0\,|\,x') = (x'\,|\,l^0m^0)^* \\ \sqrt{\mathcal{A}_r^0}^* = -i\sqrt{\mathcal{A}_r^0}.\end{aligned}\right\}(89)$$

In the centre of gravity system, i. e. for $\boldsymbol{K}^0 = 0$, these relations are obviously true, since the functions $(x'\,|\,l^0m^0)$ in this

case are ordinary spherical harmonics $Y_{l^0 m^0}(x') = P_{l^0}(\xi') \frac{e^{im^0\varphi'}}{\sqrt{2\pi}}$ and A_r^0, by (67), is given by

$$A_r^0 = \frac{4i}{|k_r^0| E_r}. \tag{90}$$

The second equation (89) then follows immediately from the definitions (52), (53) of the square root. Using the connection (I, 175) between the transformation functions $(x' | l^0 m^0)$ in two different Lorentz frames of reference as well as the general definition (62), (11), and (6) for A_r^0, we may easily prove the equation (89) to hold in an arbitrary Lorentz frame.

A comparison of (88) with (81) then shows that the asymptotic expression of ψ_r^* and Φ_r deviate only by a factor $i(-1)^{l^0} N_r^*$, i. e.

$$\psi_r^* = i(-1)^{l^0} N_r^* \cdot \Phi_r \tag{91}$$

for $r' \to \infty$. But, since both ψ_r^* and Φ_r are solutions of the conjugate complex Schrödinger equation, (91) must hold for all values of r'. For $W' \prec J_1$ we thus from (91) and (87) get

$$\psi_r^* = \frac{-i(-1)^{l^0} N_r^*}{2\pi i (W' - E_r)} (r | SU^\dagger | W' x'), \tag{92}$$

and from (80) and (92)

$$(W' x' | U | r)^* = i(-1)^{l^0} (r | SU^\dagger | W' x') \tag{93}$$

for $W' \prec J_1$.

4. A New General Condition for the Characteristic Matrix.

Consider the General equation (I, 20), which holds for any form of the Hamiltonian. By means of (I, 15) and (I, 21) and by the definitions (24) and (25) the equation (I, 20) may be written in a (W, β)-representation as follows:

$$(W'\beta'|U+U^\dagger|W^0\beta^0)+\int_{C_-(W')}\frac{(W'\beta'|U^\dagger|W''x'')\,dW''dx''(W''x''|U|W^0\beta^0)}{2\pi i(W'-W'')}$$

$$+\int_{C_+(W^0)}\frac{(W'\beta'|U^\dagger|W''x'')\,dW''dx''(W''x''|U|W^0\beta^0)}{2\pi i(W''-W^0)}=0, \tag{94}$$

or

$$(W'\beta'|U+U^\dagger|W^0\beta^0)+$$

$$+(W'-W^0)\int_{[C_+(W^0),\,C_-(W')]}\frac{(W'\beta'|U^\dagger|W''x'')\,dW''dx''(W''x''|U|W^0\beta^0)}{2\pi i(W'-W'')(W''-W^0)}=0. \tag{95}$$

In the last integral the path of integration in the W''-plane must be a curve which is simultaneously a $C_+(W^0)$-curve and a $C_-(W')$-curve, i. e. the points W^0 and W' must lie to the left and to the right of the path, respectively. In the equation (95) as it stands we cannot, therefore, let $W'\to W^0$, and in a subsequent integration over W' the point W^0 must lie to the left of the path of integration. Thus, if (95) is divided by $2\pi i(W'-W^0)$, the matrix elements of $U+U^\dagger$ will be multiplied by the function $\delta_+(W'-W^0)$. Hence, by (24), (25), (I, 15), and (I, 21), we get

$$(W'\beta'|T+T^\dagger+T^\dagger T|W^0\beta^0)=0, \tag{96}$$

or by (I, 10)

$$(W'\beta'|\Psi^\dagger\Psi|W^0\beta^0)=(W'\beta'|1|W^0\beta^0). \tag{97}$$

This is a new and very much simpler derivation of the equations (61) and (62) in I, holding now for all values of W' and W^0 inside Ω. For W' and W^0 inside the original interval J_1 the equation (97) may be written

$$\int_{J_1}(W'\beta'|\Psi^\dagger|W''x'')\,dW''dx''(W''x''|\Psi|W^0\beta^0)=$$
$$=\delta(W'-W^0)\,\delta_{\beta'\beta^0}. \tag{98}$$

It then simply expresses the orthogonality and normalization conditions for the continuous eigenfunctions

$$\left.\begin{aligned} \Psi_{E^0\beta^0}(W''x'') &= (W''x''\,|\,\Psi\,|\,E^0\beta^0) \\ \Psi_{E'\beta'}(W''x'')^* &= (E'\beta'\,|\,\Psi^\dagger\,|\,W''x'') \end{aligned}\right\} \quad (99)$$

of the Hamiltonian.

We shall now see that the equation (95) (or (97)) contains the orthogonality conditions (I, 72) for the discrete eigenfunctions, too. Let us multiply (95) by $\dfrac{S'}{W'-W^0}$ and afterwards let $W' \to W_n(l') = E_r$, $W^0 \to E^0$, where E_r is one of the values of $W' \prec J_2$ which make $S' = 0$, while $E^0 \prec J_1$. Then we get

$$\left.\begin{aligned} &\frac{(r\,|\,SU^\dagger\,|\,E^0\beta^0)}{E'_r - E^0} + \\ &\int_{I_1} \frac{(r\,|\,SU^\dagger\,|\,W''x'')\,dW''dx''\,\delta_+(W''-E^0)\,(W''x''\,|\,U\,|\,E^0\beta^0)}{E_r - W''} = 0 \\ &r = (n, l', m'). \end{aligned}\right\} \quad (100)$$

By (92), (99), and (14) this may be written

$$\int_{J_1} \Psi_r^*(W'x')\,dW'dx'\,\Psi_{E^0\beta^0}(W'x') = 0, \quad (101)$$

which is the second equation (I, 72). Further, since

$$\int_{[C_+(W^0),\,C_-(W')]} dW'' = \int_{C_-(W^0,\,W')} dW'' + \int_{C(W^0)} dW'', \quad (102)$$

where $C_-(W^0, W')$ is a path with both W^0 and W' to the right and $C(W^0)$ is a contour encircling the point W^0 in the counterclockwise sense, the equation (95) after multiplication with S', may be written

$$\left.\begin{aligned} &S'(W'\beta'\,|\,U\,|\,W^0\beta^0) + (W'\beta'\,|\,SU^\dagger\,|\,W^0\beta^0)\,S^0 + \\ &+ (W'-W^0)\int_{C_-(W^0,\,W')} \frac{(W'\beta'\,|\,SU^\dagger\,|\,W''x'')\,dW''dx''\,(W''x''\,|\,U\,|\,W^0\beta^0)}{2\pi i\,(W'-W'')\,(W''-W^0)} = 0. \end{aligned}\right\} \quad (103)$$

Here, we have used the relations

$$\int (W'\beta' \mid U^\dagger \mid W^0 x'')\, dx'' \,(W^0 x'' \mid U \mid W^0 \beta^0) =$$

$$= \int (W'\beta' \mid U^\dagger \mid W'' x'')\, dW'' dx'' \,(W'' x'' \mid R \mid W^0 \beta^0) =$$

$$= (W'\beta' \mid U^\dagger R \mid W^0 \beta^0) = (W'\beta' \mid U^\dagger \mid W^0 \beta^0) R^0$$

and $S^0 = 1 + R^0$ following from (23) and (26) in I.

If $W' \to E_r$ and $W^0 \to E_s$ with $E_r \neq E_s$, we have $S' \to 0$, $S^0 \to 0$, and, on account of (80) and (92), (103) becomes

$$\int_{J_1} \Psi_r^* (W'x')\, dW' dx' \Psi_s (W'x') = 0 \tag{104}$$

in accordance with the first equation (I, 72).

For $E_r = E_s$ a closer investigation is necessary. Since W^0 and W' lie on the same side of the path of integration in (103), we can here let $W' \to W^0$ and we get, first

$$\delta_{\beta', \beta^0} \cdot S^0 \left[R^0 + R^{0\dagger} (1 + R^0)\right] = 0,$$

or

$$S^{0\dagger} S^0 = 1 + R^0 + R^{0\dagger} + R^{0\dagger} R^0 = 1,$$

i. e. the equation (49). Further, if we differentiate (103) with respect to W' and afterwards let $W' \to E_r$ and $W^0 \to E_s$ with $E_r = E_s$, $r = (n', \beta') = (n', l', m')$, $s = (n^0, \beta^0) = (n^0, l^0, m^0)$, we get

$$\left.\begin{aligned} -\frac{dS(K_r, l^0)}{dK_r} \cdot \frac{E_r}{K_r} \cdot \delta_{\beta', \beta^0} = \\ = 2\pi i \int_{J_1} \frac{(r \mid SU^\dagger \mid W'x')\, dW' dx' \,(W'x' \mid U \mid S)}{(2\pi i)^2 (W' - E_r)(W' - E_s)} \end{aligned}\right\} \tag{105}$$

since $S' = S^0 = 0$ and $R^0 = -1$ for $W' = E_r = E_s = W^0$. On account of (80) and (92) this may be written

$$\left.\begin{aligned} &\int_{J_1} \psi_r^*(W'x')dW'dx'\,\psi_s(W''x') = \\ &= \frac{(-1)^{l^0} N_r^* N_r}{2\pi} \cdot \frac{E_r}{K_r} \cdot \frac{dS(K_r, l^0)}{dK_r} \cdot \delta_{rs}. \end{aligned}\right\} \quad (106)$$

Thus (106) together with (104) is identical with the first equation (I, 72) if we put

$$|N_r|^2 = 2\pi(-1)^{l^0} \cdot \frac{K_r}{E_r} \frac{1}{\dfrac{dS(K_r, l^0)}{dK_r}}, \quad r = (n, l^0, m^0). \quad (107)$$

Since $S_r = S(K_r, l^0)$ does not depend on m^0, N_r is a function of n and l^0 only. Further, since K_r, E_r and $|N_r|^2$ are real and positive quantities, $\dfrac{dS_r}{dK_r}$ must be real and satisfy the general condition

$$(-1)^{l^0} \frac{dS_r}{dK_r} = (-1)^{l^0} \cdot \frac{dS(K_r, l^0)}{dK_r} > 0. \quad (108)$$

The inequality (108) represents a *new general condition for the characteristic matrix* which holds in quantum mechanics independently of the form of the Hamiltonian and which may, therefore, be supposed to hold also in the new theory even in cases where no Hamiltonian of the system exists. The condition (108) implies that the zero points of the eigenvalues of S in the interval J_2, which are defined by (75), have the multiplicity one, i. e. in the neighbourhood of K_r the function $S(K^0, l^0)$ has the form

$$S(K^0, l^0) = a_r(K^0 - K_r), \quad (109)$$

where

$$a_r = \frac{dS_r}{dK_r} \neq 0$$

is a real positive or negative number, according as l^0 is even or odd, respectively.

If S_1 is a unitary matrix satisfying all the general conditions of a characteristic matrix, in particular the condition (108), S_1 defines a certain atomic system. The matrix $S_2 = S_1^\dagger$ is then also a unitary invariant matrix, but in general it will not satisfy the condition (108) and therefore may not always be taken as a characteristic matrix. However, if S_2 satisfies all the general conditions, it defines a new atomic system which may be called the 'adjoint' system. Since $R_2 = S_2 - 1 = S_1^\dagger - 1 = R_1^\dagger$, all cross-sections in the two adjoint systems are equal, but the two systems will have no closed stationary states which are identical, since S_1 and S_2 can have no common zero points.

By use of the equation (I, 26), the 'condition of completeness' (Vollständigkeitsrelation) (I, 74) of the eigenfunctions of the Hamiltonian may be written

$$(W'x'|\,T+T^\dagger+TT^\dagger|\,W''x'')+\sum_r \Psi_r(W'x')\,\Psi_r(W''x'')^* = 0, \quad (110)$$

where the variabies W' and W^0 are values of the original interval J_1. By means of (80), (92), and (107) the sum in (110) becomes

$$\left.\begin{aligned}\sum_r \Psi_r(W'x')\,\Psi_r(W''x'') &= \sum_r \frac{(W'x'|\,U|\,r)\,(r|\,U^\dagger|\,W''x'')}{2\pi i\,(W'-E_r)\,(E_r-W'')}\cdot\frac{S_r}{\dfrac{dS_r}{dE_r}}\\ &= \sum_{\beta^0}\int_{C(0)}\frac{(W'x'|\,U|\,W^0\beta^0)\,dW^0\,(W^0\beta^0|\,U^\dagger|\,W''x'')}{(2\pi i)^2\,(W'-W^0)\,(W^0-W'')},\end{aligned}\right\} \quad (11$$

where the path of integration $C(0)$ in the W^0-plane is a series of contours encircling in a counter-clockwise sense all the zero points (75) of S^0 in such a way that the integrand has no singularities inside $C(0)$. Here, we have used CAUCHY's theorem and the fact that S^0 in the neighbourhood of any zeropoint $W^0 = E_r$ is of the form

$$S^0 = \frac{dS_r}{dE_r}(W^0 - E_r).$$

If we multiply (110) by $2\pi i(W'-W'')$, we get, by means of (111),

$$\left.\begin{aligned}&(W'x'\,|\,U+U^{\dagger}\,|\,W''x'')+\\ &+(W'-W'')\sum_{\beta^0}\int_{[C_+(0,W'')\,C_-(W')]}\frac{(W'x'\,|\,U\,|\,W^0\beta^0)\,dW^0\,(W^0\beta^0\,|\,U^{\dagger}\,|\,W''x'')}{2\pi i\,(W'-W^0)\,(W^0-W'')}=0,\end{aligned}\right\}\quad(112)$$

where the path of integration is a C_+-curve for all the zero points and for the point W'' and, simultaneously, a C_--curve for the point W'. The equation (112) is a general condition for the matrix U, holding in quantum mechanics for any system and for all values of W' and W'' inside Ω. (112) has a similar form as the equation (95).

We shall now discuss the question how far the atomic system is defined in the quantum mechanical sence, if only the characteristic matrix S is given. Assuming that the given matrix S satisfies all the general conditions (I, 27), (I, 28), and (108), we can try to find a matrix U which satisfies the general equations (95) and (112), and which, further, is connected by the given matrix $R=S-1$ by the equation (I, 23). If we have found such a matrix U, we can define a wave matrix Ψ by (I, 10) and (I, 15). Furthermore, we can define a set of functions Ψ_r by (80), where r is an index enumerating the zero points of the eigenvalues of S, and N_r may be taken as the square root of the right hand side of (107). The functions Ψ_r together with the functions (99) are then the eigenfunctions of a 'Hamiltonian' H defined by

$$\left.\begin{aligned}&(W'x'\,|\,H\,|\,W''x'')=\\ &=\sum_{\beta^0}\int(W'x'\,|\,\Psi\,|\,E^0\beta^0)\,E^0\,dE^0\,|\,E^0\beta^0\,|\,\Psi^{\dagger}\,|\,W''x'')+\\ &\qquad+\sum_r\Psi_r\,(W'x')\,E_r\,\Psi\,(W''x'')^*\end{aligned}\right\}\quad(113)$$

with the discrete eigenvalues $E_r=W_n^0(I^0)$ given by (75) and the continuous eigenvalues E^0 in the interval J_1, i. e. $W_m^0<E^0<\infty$. If the operator (113) is used as Hamiltonian in a Schrödinger equation, we get the same result as regards the 'observable' quantities of HEISENBERG as that following directly from the given S-matrix. However, if there is a solution at all of the equations (95), (112), and (I, 23) for the matrix U, it is easily seen that there are many solutions. This means that there are

many Hamiltonians (113) which give the same result as regards the 'observable' quantities of HEISENBERG, but the wave functions corresponding to the different Hamiltonians will only be identical for large distances apart of the particles and they will, thus, in general lead to different results regarding, for instance, the probability of the particles being in small distances from each other.

To see this, let us consider the simple case where the eigenvalues of S have no zero points in J_2, which means that the system has no closed stationary states. In this case the equations (95) and (112), expressing the orthogonality properties and the completeness of the eigenfunctions, may be expressed in terms of the wave matrix Ψ by the simple matrix equations

$$\left.\begin{aligned} \Psi^\dagger \Psi &= 1 \\ \Psi \Psi^\dagger &= 1 \end{aligned}\right\} \quad (114)$$

showing that the wave matrix Ψ is a unitary matrix in this case. (Cf. (I, 62) and (I, 63)). Now, let us assume that we have found a solution of (114) of the form (I, 10)

$$\Psi = 1 + T \quad (115)$$

$$(\Psi' x' \,|\, T \,|\, W^0 x^0) = \delta_+ (W' - W^0)\,(W' x' \,|\, U \,|\, W^0 x^0), \quad (116)$$

where U is a matrix satisfying (I, 23), i. e.

$$\left.\begin{aligned} (W^0 x' \,|\, U \,|\, W^0 x^0) &= (x' \,|\, R \,|\, x^0) \\ R &= S - 1. \end{aligned}\right\} \quad (117)$$

The Hamiltonian (113) is then given by the matrix equation

$$H = \Psi W \Psi^\dagger. \quad (118)$$

However, the matrix

$$\breve{\Psi} = \Psi A \quad (119)$$

with

$$A^\dagger A = A A^\dagger = 1 \quad (120)$$

will also satisfy (114), i. e.

$$\breve{\Psi}^\dagger \breve{\Psi} = \breve{\Psi} \breve{\Psi}^\dagger = 1. \quad (121)$$

Putting

$$A = 1 + B \tag{122}$$

we get from (120) for B

$$\left.\begin{aligned} B + B^\dagger + B^\dagger B &= 0 \\ B^\dagger B &= B B^\dagger . \end{aligned}\right\} \tag{123}$$

Since the new wave matrix must have the same form (115), (116) as the old one we have

$$\breve{\Psi} = 1 + \breve{T} \tag{115'}$$

$$(W'x' \,|\, \breve{T} \,|\, W^0x^0) = \delta_+ (W' - W^0)\,(W'x' \,|\, \breve{U} \,|\, W^0x^0). \tag{116'}$$

Thus, from (115'), (119), (122), and (115), we get

$$\breve{T} = \breve{\Psi} - 1 = \Psi A - 1 = \Psi - 1 + \Psi B = T + B + TB. \tag{124}$$

On account of (31) we then get from (116'), (116), and (25),

$$\left.\begin{aligned} &(W'x' \,|\, \breve{U} \,|\, W^0x^0) = \\ &= (W'x' \,|\, U \,|\, W^0x^0) + 2\pi i (W' - W^0)\,(W'x' \,|\, B \,|\, W^0x^0) + \\ &+ 2\pi i (W' - W^0) \int_{C_-(W')} \frac{(W'x' \,|\, U \,|\, W''x'')\, dw''\, dx''\, (W''x'' \,|\, B \,|\, W^0x^0)}{2\pi i (W' - W'')} \end{aligned}\right\} \tag{125}$$

Now, we are only interested in those matrices U which satisfy also the equation (117) with the given R. This means that the matrix elements of $\breve{U}$ and U must be equal for $W' = W^0$. From (125) this is seen to be the case if $(W'x' \,|\, B \,|\, W^0x^0)$ has no singularity for $W' = W^0$.

Thus, if B is any matrix whose matrix elements are finite for $W' = W^0$ and which satisfies (123), the matrix $\breve{U}$ defined by (125) will satisfy the conditions (95), (112), and (I, 23) provided that U is a solution of these equations, and the Hamiltonian

$$\breve{H} = \breve{\Psi}\, W\, \breve{\Psi}^\dagger \tag{118'}$$

will give the same results as regards HEISENBERG's 'observable' quantities as the Hamiltonian H defined by (118). This means that for a given characteristic matrix S we can either define a

large number of different Hamiltonians or we can find no Hamiltonian at all for the system considered.

5. Non-Stationary States. New General Conditions for S. Determination of the Decay Constant of Radioactive Systems by means of the Characteristic Matrix.

Since the function $\psi_{W^0\beta^0}(W'x')$ defined by (21) satisfies the time independent Schrödinger equation (22) for all values of W^0 inside Ω. the function

$$\psi = \psi_{W^0\beta^0} e^{-iW^0 t} \tag{126}$$

satisfies the time dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi \tag{127}$$

for all values of W^0 inside Ω. However, for complex values of W^0 the function (126) does not correspond to a stationary state. If W^0 lies in the lower half plane inside Ω, we may write

$$W^0 = E_1 - i\frac{\lambda}{2} \tag{128}$$

with E_1 and λ real and positive, and the time factor in (126) takes the form

$$e^{-iE_1 t - \frac{\lambda}{2} t},$$

which means that the amplitude of the wave function (126) decreases exponentially. The asymptotic expression for the wave function (126) is obtained from (65) by multiplication with $e^{-iW^0 t}$. In the centre of gravity system we get for this asymptotic expression

$$C\left[\frac{e^{-ir'k^0}}{r'}(-1)^{l^0+1} + \frac{e^{ir'k^0}}{r'}S^0\right] Y_{l^0m^0} e^{-iW^0 t}, \tag{129}$$

where C and k^0 are constants determined by (67), i. e.

$$k^0 = \frac{1}{2}\sqrt{E_1^2 - 4K^2 - \left(\frac{\lambda}{2}\right)^2 - iE_1\lambda} = k_1^0 - ik_2^0 \tag{130}$$

with k_1^0 and k_2^0 real and positive for $\lambda > 0$. The first term in (129) corresponds to an ingoing spherical wave in the relative coordinate space and the corresponding current density in the direction of increasing r' is proportional to $-2\,k_1^0\,e^{-2k_2^0 r'}\,|Y_{l^0 m^0}|^2$. Similarly, the second term in (129) is an outgoing spherical wave with the corresponding current density equal to $2\,k_1^0\,e^{2k_2^0 r'}\,|Y_{l^0 m^0}|^2$. From this it follows at once that

A the eigenvalues of S, i.e. S^0, cannot have any zero points for $W^0 \prec \Omega$ in the lower half plane, where E_1 and λ are positive,

for this would correspond to a state in which we have an ingoing current through the surface of a large sphere while, simultaneously, the total probability of the system, having an r' smaller than the radius of the sphere, decreases exponentially with time, the rate of decrease being determined by the damping constant λ. But this would obviously be in contradiction with the continuity equation following from the Schrödinger equation for any form of the Hamiltonian. Thus, the statement A represents a further general condition for the matrix S.

If we multiply (126) by $S^{0\dagger} = (S^0)^{-1}$, we get a solution of the Schrödinger equation with the asymptotic expression

$$C\left[\frac{e^{-ir'k^0}}{r'}(-1)^{l^0+1}S^{0\dagger}+\frac{e^{ir'k^0}}{r'}\right]Y_{l^0 m^0}\,e^{-iW^0 t} \qquad (131)$$

Considering now a value of $W^0 \prec \Omega$ in the lower half plane for which $S^{0\dagger} = 0$, i. e. where S^0 has a singular point, the wave function (131) corresponds to a radioactive decay process, since we have then an outgoing wave only, for large values of r'. Thus, *if S^0 has singular points inside Ω in the lower half of the W^0-plane, this means that our system may undergo a radioactive disintegration. The decay constant λ for this process is equal to twice the numerical value of the imaginary part of the value of W^0 for which S^0 is singular, and the real part E_1 of W^0 may be interpreted as the energy of the decaying system.* Of course the energy of the system is only defined with a definite uncertainty given by the constant λ which also determines the breadth of the energy level.

Considering now a value of $W^0 \prec \Omega_+$, i.e. in the upper half of the W^0-plane, where

$$W^0 = E_1 + i\frac{\lambda}{2}, \quad E_1 > 0, \quad \lambda > 0 \tag{132}$$

we find from (131) in a similar way as before

B $S^{0\dagger}$ *has no zero points, i. e.* S^0 *has no singular points for* $W^0 \prec \Omega$ *in the upper half of the* W^0*-plane, where the real and imaginary parts of* W^0 *are positive*;

for this would contradict the continuity equation. Further, introducing (132) into (129), we see that the zero points of S^0 in the upper half plane correspond to states where we have ingoing spherical waves only, and an exponential increase in time of the probability of finding the particles in distances r' smaller than a given value. These states thus correspond to processes which are the reverse of a decay process, and the imaginary parts of W^0 in such cases determine the probabilities of capture processes.

Throughout this section we have worked in the centre of gravity system where the total momentum is zero. Let us now consider a decaying system from the point of view of an observer in a Lorentz system, where the total momentum is $\boldsymbol{K}^0 \neq 0$. If $\boldsymbol{v}$ is the velocity of the centre of gravity system relative to the Lorentz system of the observer, the total momentum $\boldsymbol{K}^0$ and energy E_1 are connected with the energy $\overline{E}_1$ in the centre of gravity system by the equations

$$\boldsymbol{K}^0 = \frac{\overline{E}_1 \cdot \boldsymbol{v}}{\sqrt{1 - v^2}}, \quad E_1 = \frac{\overline{E}_1}{\sqrt{1 - v^2}}. \tag{133}$$

Here, we have assumed the decay constant to be so small compared with the energy that the system has practically a well defined energy.

Now, let

$$K^0 = \sqrt{W^{0\,2} - |\boldsymbol{K}|^2} = \overline{W}^0 \tag{134}$$

be a complex value of the invarient K^0 for which the invariant function $S^0 = S(K^0, l^0)$ is infinite. If we put $W^0 = E_1 - i\frac{\lambda}{2}$ and and $\overline{W}^0 = \overline{E}_1 - i\frac{\overline{\lambda}}{2}$ the quantities λ and $\overline{\lambda}$ represent the decay constants of our system in the two Lorentz systems of reference,

and the equation (134) gives, by means of (133), a relation between λ and $\bar{\lambda}$. To be consequent, we have here to neglect all terms of the order λ^2 and $\bar{\lambda}^2$, thus we get, if (134) is squared,

$$E_1^2 - i\lambda E_1 - \frac{\bar{E}_1^2 v^2}{1-v^2} = \bar{E}_1^2 - i\bar{\lambda}\bar{E}_1 .$$

The real part of this equation gives just the second equation (133), while the imaginary part yields

$$\lambda E_1 = \bar{\lambda}\bar{E}_1$$

i. e.

$$\lambda = \bar{\lambda}\cdot\sqrt{1-v^2},$$

which leads to the well-known formula

$$\tau = \frac{\bar{\tau}}{\sqrt{1-v^2}}, \tag{135}$$

for the lifetimes of the system in the two Lorentz frames.

6. Examples Illustrating the General Theory.

In all cases, where the system considered has a Hamiltonian, the characteristic matrix is determined by the solution of the Schrödinger equation and, from (65), we see that the eigenvalues S^0 of S may be obtained from the asymptotic expressions for the continuous stationary solutions. In the simple case of a non-relativistic two particle system, we have in the centre of gravity system of reference the following differential equation for the radial part $\frac{\chi(r)}{r}$ of the wave function in the relative coordinate space

$$\frac{d^2\chi}{dr^2} + \left(k^{0\,2} - \kappa V(r) - \frac{l^0(l^0+1)}{r^2}\right)\chi = 0, \tag{136}$$

where $V(r)$ is the interaction potential. k^0 is the relative momentum given by (67)

$$k^0 = \frac{1}{2}\sqrt{W^{0\,2} - (2\kappa)^2}, \tag{137}$$

which, in the non-relativistic approximation considered here, reduces to

$$k^0 = \sqrt{K\,\varepsilon} \tag{138}$$

with

$$W^0 = 2K + \varepsilon. \tag{139}$$

Now, let us assume that the potential V is zero for $r > R$, where R is a finite, but possibly very large distance. In the region $r > R$ the solution $\chi_{II}(r)$ of the equation (136) is given by

$$\chi_{II} = \varphi^{(2)} + \varphi^{(1)} S^0 \tag{140}$$

with

$$\left.\begin{aligned} \varphi^{(1)} &= i^{l^0+1}\sqrt{\frac{\pi}{2}k^0 r}\cdot H^{(1)}_{l^0+\frac{1}{2}}(k^0 r), \\ \varphi^{(2)} &= i^{l^0+1}\sqrt{\frac{\pi}{2}k^0 r}\cdot H^{(2)}_{l^0+\frac{1}{2}}(k^0 r). \end{aligned}\right\} \tag{141}$$

Here $S^0 = S(W^0, l^0)$ is an eigenvalue of S which, in the centre of gravity system where $K^0 = W^0$, is a function of W^0 and l^0 only. Further, $H^{(1)}$ and $H^{(2)}$ are the Hankel functions of the first and second kind, respectively. χ_{II} is obviously a solution of (136) for $r > R$ and the asymptotic expression of χ_{II} for $r \to \infty$ has the right form

$$e^{-ik^0 r}(-1)^{l^0+1} + e^{ik^0 r} S^0, \tag{142}$$

as is seen if we use the known asymptotic expresssions for the Hankel functions. In the region $r < R$, our solution may be written

$$\chi_I = C\cdot\chi(r), \tag{143}$$

where $\chi(r)$ is a solution of (136) which is zero for $r = 0$, and C is a constant.

The condition that our solution and its first derivative must be continuous for $r = R$ gives us at once two equations from which we can determine the constants C and S^0. For S^0 we get

$$S^0 = S(W^0, l^0) = -\frac{\varphi^{(2)}(R)\,\chi'(R) - \varphi^{(2)\prime}(R)\,\chi(R)}{\varphi^{(1)}(R)\,\chi'(R) - \varphi^{(1)\prime}(R)\,\chi(R)}. \tag{144}$$

Let us now consider the simple case where the potential is a negative constant for $r < a$ and zero for $r > a$, i. e.

$$V = \begin{cases} -V_0 = \text{constant} & \text{for } r < a \\ 0 & \text{for } r > a. \end{cases} \tag{145}$$

In this case, we may put $R = a$ and the function χ in (143) is simply

$$\chi(r) = \sqrt{\frac{\pi}{2} k_I^0 r} \cdot J_{l^0+\frac{1}{2}}(k_I^0 r), \tag{146}$$

where $J_{l^0+\frac{1}{2}}$ is a Bessel function of the first kind and

$$k_I^0 = \sqrt{\kappa V_0 + k^{0^2}} \tag{147}$$

If we introduce (146) and (141) into (144), we get a general expression for the eigenvalues $S^0 = S(W^0, l^0)$ and, from this, we easily obtain the matrix elements of S in a (W, x)-representation. Omitting the factor $\delta(W' - W^0)$ we get for these matrix elements

$$(x' \mid S \mid x^0) = \sum_{l^0 m^0} (x' \mid l^0 m^0) S^0 (l^0 m^0 \mid x^0), \tag{148}$$

which directly determine all scattering cross-sections. Since

$$J^{\dagger}_{l^0+\frac{1}{2}} = J_{l^0+\frac{1}{2}}, \quad H^{(1)\dagger}_{l^0+\frac{1}{2}} = H^{(2)}_{l^0+\frac{1}{2}} \text{ and } H^{(2)\dagger}_{l^0+\frac{1}{2}} = H^{(1)}_{l^0+\frac{1}{2}},$$

we see that the function S^0, defined by (144), (141), and (146), satisfies the equation (49), i. e.

$$S^{0\dagger} = \frac{1}{S^0}.$$

For $l^0 = 0$ we get from (141) and (146)

$$\left.\begin{aligned} \chi &= \sin k_I^0 r \\ \varphi^{(1)} = e^{ik^0 r}, \quad \varphi^{(2)} &= -e^{-ik^0 r} \end{aligned}\right\} \tag{149}$$

and, thus, by (144)

$$S^0 = S(W^0, 0) = \frac{1 + \frac{ik^0}{k_I^0} \operatorname{tg} k_I^0 a}{1 - \frac{ik^0}{k_I^0} \operatorname{tg} k_I^0 a} \cdot e^{-2ik^0 a} \tag{150}$$

To get the energy values of the system in the closed stationary states corresponding to $l^0 = 0$, we have to determine the

zero points of S^0 in (150) for $W^0 < 2K$, i. e. for $\varepsilon < 0$. Since we have, in this case, $k^0 = -i|k^0| = -i\sqrt{-K\varepsilon}$ the zero points are determined by the equation

$$1 + \frac{|k^0|}{k_1^0} \operatorname{tg} k_1^0 a = 0. \tag{151}$$

The values of W^0 or ε, given by (151), are equal to the discrete eigenvalues of the Hamiltonian following from the Schrödinger equation.

It is easily verified that the function (150) satisfies the general condition (108) and the conditions A and B on p. 35 and p. 36, respectively. The function (150) has no singular points in the lower half of the W^0-plane, which means that our system cannot undergo spontaneous disintegrations. Further, we see that our system has no adjoint system, since the function $S_2^0 = S^{0\dagger}$ does not satisfy the conditions (108).

It is now possible in an infinite number of ways to define the S-matrix of a relativistic system which, in the non-relativistic approximation, is identical with the system just considered. In the centre of gravity system of reference we may, for instance, take the S-matrix defined by (150) (or, more generally, by (141), (144), and (146)) with k^0 given by (137) or with k^0 equal to any other analytic function $f(W^0)$ of W^0, which, in the non-relativistic approximation, becomes identical with (138). The only condition for $f(W^0)$ is that S must satisfy the general conditions (108) and A, B on p. 35 and p. 36. In order to get the expression for the characteristic matrix in an arbitrary frame of reference, we have then merely to replace W^0 by K^0 in the expression for S^0. For a system defined in this way, it will not in general be possible to define a Hamiltonian of the system.

In his third paper in the ZS. f. Phys., HEISENBERG[2] considered a system defined by a characteristic matrix with the eigenvalues

$$\left.\begin{aligned} S(K^0, 0) &= \frac{1 + \frac{i\alpha}{K} k^0}{1 - \frac{i\alpha}{K} k^0} \\ S(K^0, l^0 > 0) &= 1, \end{aligned}\right\} \tag{152}$$

where α is a constant, and

$$k^0 = f(K^0) = \varkappa \sqrt{1 - \frac{4\,\varkappa^2}{K^{0\,2}}}. \tag{153}$$

It is easily seen that this system may be obtained as a limiting case from the system defined by (144), (141), and (146) with $k^0 = f(K^0)$ given by (153). For, if $a \to 0$ and $V^0 \to \infty$ in such a way that

$$\lim \frac{\operatorname{tg}(\sqrt{\varkappa\, V_0}\, a)}{\sqrt{\varkappa\, V_0}} = \alpha,$$

we just get the expressions (152) for the eigenvalues of the characteristic matrix. The system defined by (152) has only one bound state if α is negative, and for the rest mass of the system in this state we get from (152) and (153)

$$K^0 = \frac{2\,\varkappa}{\sqrt{1 + \frac{1}{\alpha^2}}} \tag{154}$$

Let us now consider the case of a non-relativistic system of two particles interacting according to the Coulomb law. Since our formalism applies only to cases where the potential goes to zero faster than $\frac{1}{r}$ as r tends to infinity, we shall assume the Coulomb potential to break off at a large distance R, i. e.

$$V(r) = \begin{cases} \dfrac{e_1\, e_2}{r} & \text{for } r < R \\ 0 & \text{for } r > R \end{cases} \tag{155}$$

where e_1 and e_2 are the charges of the two particles. The eigenvalues of S are again determined by (144) and (141), where $\chi(r)$ is the ordinary solution of the Schrödinger equation for the Coulomb case. If R is chosen sufficiently large, we may use the asymptotic expressions for $\varphi^{(1)}$, $\varphi^{(2)}$ and χ in (144) and we get[4]

$$S^0 = S(W^0, l^0) = (2\,k^0 R)^{-\frac{2\,i}{k^0 a}} \frac{\Gamma\left(l^0 + 1 + \frac{i}{k^0 a}\right)}{\Gamma\left(l^0 + 1 - \frac{i}{k^0 a}\right)} \tag{156}$$

where k^0 is given by (138) and (139), and

$$a = \frac{2}{K e_1 e_2} . \tag{157}$$

The closed stationary states are determined by the zero points of (156) for $k^0 = -i|k^0|$, i. e. by the equation

$$l^0 + 1 + \frac{1}{|k^0| a} = -\nu, \quad \nu = 0, 1, 2, \cdots . \tag{158}$$

Thus, we have only closed stationary states if a is negative, i. e. in the case of attraction. Putting $n = \nu + l^0 + 1 \geq l^0 + 1$, we get for the energy ε the formula

$$\varepsilon^0 = -\frac{1}{K a^2 n^2}, \tag{159}$$

which gives BOHR's formula for the energies of a Coulomb system if we introduce the ordinary units. The condition (108) as well as the conditions A and B in Section 5 are easily seen to be satisfied by the expression (156). The adjoint system is, in this case, simply a system in which the sign of a is reversed, i. e. where we have repulsion instead of attraction. All cross sections are identical in these two cases, but in the case of positive a we have, of course, no bound states.

As in the case of a rectangular potential well, we also here get a relativistic generalization of the non-relativistic Coulomb system by replacing k^0 in (156) by an arbitrary analytic function $f(W^0)$ which, in the non-relativistic approximation, reduces to (138), (139). In order to determine the function $f(W^0)$ we need some information about the scattering of charged particles in the relativistic region. Now, in the approximation where e_1, e_2 may be treated as small, the cross section for the scattering of fast charged particles may be determined in a relativistically invariant and unambiguous way by a simple correspondence treatment[5]. The condition that our S-matrix shall give this scattering cross section in the limit of small $e_1 e_2$ uniquely determines the function $f(W^0)$. The S-matrix determined in this way then represents a relativistic system which, in the non-relativistic region of velocities, corresponds to a Coulomb po-

tential and which, for large velocities but small values of $e_1 e_2$, gives the right scattering cross section as determined by the correspondence principle.

Finally, we shall consider a non-relativistic two particle-system with the potential

$$V = \begin{cases} 0 & \text{for } r < a \\ U = \text{const.} & \text{for } a < r < a + l \\ 0 & \text{for } r > a + l. \end{cases} \tag{160}$$

For $l^0 = 0$, we in this case get from (144)

$$\left.\begin{aligned} S^0 &= S(W^0, l^0 = 0) = \\ &= e^{-2ik^0(a+l)} \frac{(\gamma+1)(ik_I^0 + ik^0) - (ik_I^0 - ik^0)\, e^{-2ik_I^0 l}}{(\gamma+1)(ik_I^0 - ik^0) - (\gamma-1)(ik_I^0 + ik^0)\, e^{-2ik_I^0 l}}. \end{aligned}\right\} \tag{161}$$

Here,

$$\gamma = \frac{ik_I^0}{k^0} \operatorname{tg} k^0 a, \tag{162}$$

k^0 is given by (138), (139), and

$$k_I^0 = \sqrt{k^{02} - \varkappa U}. \tag{163}$$

For real values of $W^0 < W_m^0$, i. e. for $k^0 = -i|k^0|$, S^0 has no zero points, i. e. the system has no closed stationary states. But S^0 is singular for certain complex values of W^0 in the lower half of the W^0-plane, i. e. for

$$\left.\begin{aligned} W^0 &= E - i\frac{\lambda}{2} \\ E &= 2\varkappa + \varepsilon > 0, \quad \lambda > 0. \end{aligned}\right\} \tag{164}$$

This indicates that the system has radioactive states in which the system disintegrates. According to the general theory in Section 5, these states are determined by the singular points of (161), i. e. by the equation

$$\gamma + 1 = (\gamma - 1)\,\frac{ik_{\mathrm{I}}^{0} + ik^{0}}{ik_{\mathrm{I}}^{0} - ik^{0}}\,e^{-2ik_{\mathrm{I}}^{0}l}. \tag{165}$$

(165) is a complex equation from which we can determine the real and the imaginary part of W^0, i. e. E and λ. For $U > \varepsilon$, the exponential on the right hand side contains a real factor $e^{-2\sqrt{\kappa(U-\varepsilon)}\,l}$ which in all practical cases is a very small quantity. This makes the solution of (165) very easy to perform. For the real part E of W^0, we get the equation

$$\operatorname{tg}(\sqrt{\kappa\varepsilon}\,a) = -\sqrt{\frac{U-\varepsilon}{\varepsilon}} \tag{166}$$

and for the imaginary part

$$\lambda = \frac{16\,\varepsilon^{\frac{3}{2}}(U-\varepsilon)^{\frac{3}{2}}}{U^2\left(1 + a\sqrt{\kappa(U-\varepsilon)}\right)}\,e^{-2\sqrt{\kappa(U-\varepsilon)}\,l}. \tag{167}$$

The equation (166) determines the energy values E of the system in the radioactive states, while (167) gives the relation between the decay constant λ and the energy. Both formulae are, of course, in agreement with the results obtained from the theory of GAMOW and CONDON and GURNEY for the potential (160)[6]. The expression (161) is in accordance with the conditions A and B in Section 5.

Conclusion.

In the present paper we have investigated in detail only simple systems consisting of two particles with no possibilities for creation and annihilation processes. It has been shown that the closed stationary states may be obtained by analytic continuation of the functions $S^0 = S(W^0, \beta^0)$ representing the 'eigenvalues' of the characteristic matrix in a representation where the variables (W, β) of the complete set of collision constants are on diagonal form. In this analytic continuation W^0 is considered a complex variable, while the eigenvalues β^0 of the other collision constants are regarded as real variables.

If S^0 has any zero points for real values of W^0 smaller than the minimum value W_m^0 of the total kinetic energy, the system considered has closed stationary states with energies equal to the values of W^0 in these zero points, i. e. the energy values are determined by the equation (75). These zero points have the multiplicity one in accordance with the general condition (108).

Further more, if S^0 has any poles in the lower half of the W^0-plane, this indicates that the system has radioactive states with energies and decay constants determined by the real and imaginary parts of the complex values of W^0 in these singular points. Finally, the functions S^0 satisfy the conditions A and B in Section 5, stating that S^0 has no zero points in the lower half plane and no singular points in the upper half plane inside the region Ω where S^0 is analytic.

The results obtained for two particle systems in this paper may be supposed to hold also in the general case of a many particle system with possibilities for creation and annihilation processes, the only difference probably being that the number of collision constants β which together with W constitute a complete set is then larger than in the case of a simple two particle system.

Added in proof: After completion of the present paper, S.T.MA[7] has published a note in which he points out that the condition (75) for the energy values in closed stationary states is only a necessary condition, i. e. that, in some cases, S^0 may have zero points which do not correspond to closed stationary states. A subsequent investigation by D. TER HAAR[8] shows that, anyway in the case considered by MA, the condition (108) may be used in order to discard these redundant zeros.

References.

1) C. MØLLER, General Properties of the Characteristic Matrix in the Theory of Elementary Particles. I. D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. XXIII, Nr. 1 (1945).
2) Cf. W. HEISENBERG, Die "beobachtbaren Grössen" in der Theorie der Elementarteilchen. III. ZS. f. Phys. in (1944).
3) W. HEISENBERG, *loc. cit.* 2).
4) W. GORDON, Ueber den Stoss zweier Punktladungen nach der Wellenmechanik. ZS. f. Phys. **48**, 180 (1928).
5) C. MØLLER, Ueber den Stoss zweier Teilchen unter Berücksichtigung der Retardation der Kräfte. ZS. f. Phys. **70**, 786 (1931).
6) See J. KUDAR, Zur Quantenmechanik der Radioactivität. ZS. f. Phys. **53**, 96 (1929) and C. MØLLER, ZS. f. Phys. **55**, 451 (1929). The equation in the middle of p. 466 of the last mentioned paper is identical with the formula (167) in the present paper.
7) S. T. MA, Phys. Rev. **69**, 668 (1946).
8) D. TER HAAR, On the Redundant Zeros in the Theory of the Heisenberg Matrix. Physica, to appear presently.

Indleveret til Selskabet den 17. April 1946.
Færdig fra Trykkeriet den 14. Oktober 1946.